Europ. J. Combinatorics (1981) **2**, 21–26

Cyclic Block Designs With Block Size 3

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We prove that the necessary conditions for the existence of cyclic block designs with block size 3 are sufficient with two exceptions.

1. HISTORICAL BACKGROUND

Following Hanani [8] we denote by $B[k, \lambda; v]$ a balanced incomplete block design with v elements, block size k, and balance factor λ. We denote by $B(k, \lambda)$ the set $\{v | B[k, \lambda; v]$ exists$\}$.

In 1847, Kirkman [13] determined the set $B(3, 1)$ (see also Steiner [23] and Reiss [20]). Bhattacharya [2] used techniques suggested by Bose [3] to completely determine $B(3, 2)$; Skolem [17] determined $B(3, 3)$. $B(3, \lambda)$ for every λ was determined by Hanani [7].

THEOREM 1. $v \in B(3, \lambda)$ *if and only if*
(i) $\lambda \equiv 1, 5 \pmod 6$ *and* $v \equiv 1, 3 \pmod 6$ *or*
(ii) $\lambda \equiv 2, 4 \pmod 6$ *and* $v \equiv 0, 1 \pmod 3$ *or*
(iii) $\lambda \equiv 3 \pmod 6$ *and* $v \equiv 1 \pmod 2$ *or*
(iv) $\lambda \equiv 0 \pmod 6$ *and* $v \geqslant 3$.

Hanani's proof employs recursive construction techniques; direct proofs have been given by Nash-Williams [15] and Hwang and Lin [12].

This work is naturally only a preface to the vast body of literature on block designs with block size 3. We refer the reader to the excellent bibliography of Doyen and Rosa [5] on Steiner systems for a guide to the work on $B[3, 1; v]$ designs. The existence of $B[3, 2; v]$ designs with additional constraints has been examined by Mendelsohn and others [1, 11, 14].

In this paper, we completely resolve the existence of *cyclic* $B[3, \lambda; v]$ designs. A *cyclic block design* $CB[k, \lambda; v]$ is a $B[k, \lambda; v]$ with elements $\{0, \ldots, v-1\}$ for which if $\{a_1, \ldots, a_k\}$ is a block, $\{a_1+1, \ldots, a_k+1\}$ is also a block (addition performed modulo v). We further denote by $CB(k, \lambda)$ the set $\{v | CB[k, \lambda; v]$ exists$\}$.

For a block $b = \{a_1, \ldots, a_k\}$, define the set $CL(b) = \{\{a_1+i, \ldots, a_k+i\} | 0 \leqslant i < v$, addition mod $v\}$. A *collection of starter blocks* for a $CB[k, \lambda; v]$ with the multiset of blocks B is a multiset $S \subseteq B$ for which the multiset $\{b | b \in CL(s), s \in S\} = B$.

Now restrict attention to $CB[3, \lambda; v]$. Each block b has $|CL(b)| = v/3$ or v. In the former case, the block is called *short* and further it belongs to $CL(\{0, v/3, 2v/3\})$. Finding a $CB[3, \lambda; v]$ is equivalent to finding a suitable collection of starter blocks. This problem can be recast as follows. Consider a collection of starter blocks; each starter block $s = \{a, b, c\}$ is represented as the collection of six differences $\{a-b, b-a, a-c, c-a, b-c, c-b\}$. To represent this set, it suffices to retain only the *difference triple* for the starter block, which is $\{\min(a-b, b-a), \min(a-c, c-a), \min(b-c, c-b)\}$. Let $\{x, y, z\}$ be a difference triple obtained in this manner. It is evident that either *xy and z* sum to v, or one is the sum of the other two. It is further the case that none of x, y or z exceeds $v/2$. A *difference triple* is taken to be a triple satisfying these properties.

0195-6698/81/010021+06 \$01.00/0

In 1897, Heffter [9] posed two "difference problems":

HEFFTER'S DIFFERENCE PROBLEM I. Can one partition the set $\{1, \ldots, (v-1)/2\}$ into difference triples?

HEFFTER'S DIFFERENCE PROBLEM II. Can one partition the set $\{1, \ldots, v/3-1, v/3+1, \ldots, (v-1)/2\}$ into difference triples?

Heffter observed that a solution to his first difference problem would give a solution to the existence of $CB[3, 1; v]$ for $v \equiv 1 \pmod 6$. Further he noted that a solution to his second difference problem (allowing for the inclusion of the short starter block $\{0, v/3, 2v/3\}$) would give a solution to the existence of $CB[3, 1; v]$ for $v \equiv 3 \pmod 6$.

In the spirit of Heffter's difference problems, we pose generalized versions for arbitrary λ. $D(v, \lambda)$ denotes the multiset containing each i for $0 \leqslant i < v/2$ λ times when v is odd. When v is even, $D(v, \lambda)$ contains in addition the difference $v/2$ $\lambda/2$ times (thus $D(v, \lambda)$ is not defined for v even and λ odd; this will create no difficulties).

There are then two generalized difference problems:

I. if $v \not\equiv 0 \pmod 3$, can $D(v, \lambda)$ be partitioned into difference triples?

When $v \equiv 0 \pmod 3$, we define $D_0(v, \lambda) = D(v, \lambda)$ and $D_m(v, \lambda) = D_{m-1}(v, \lambda) - \{v/3\}$. The second problem is then:

II. If $v \equiv 0 \pmod 3$, is there an m for which $D_m(v, \lambda)$ can be partitioned into difference triples?

The resolution of these two difference problems is equivalent to a complete determination of $CB(3, \lambda)$, as the reader can easily verify.

2. NECESSARY CONDITIONS

It is evident that $CB(3, \lambda) \subseteq B(3, \lambda)$. The cyclic constraint imposes further necessary conditions.

LEMMA 2. *If $v \equiv 2 \pmod 4$ and $\lambda \equiv 2 \pmod 4$, $v \notin CB(3, \lambda)$.*

PROOF. The fundamental observation is this: since v is even, every difference triple uses either zero or two odd differences. Now $D(v, \lambda)$ contains an odd number of odd differences; in fact, for $v = 4m+2$, it contains $2\lambda m + \lambda/2$ odd differences—this is odd since $\lambda/2$ is odd. This completes the proof when $v \not\equiv 0 \pmod 3$. In the case $v \equiv 0 \pmod 3$, $v = 12m+6$. But then the difference used by the short block(s) is $4m+2$ which is even. Hence the difference triples must use an odd number of odd differences and this cannot be.

Theorem 1 and Lemma 2 give us the following necessary condition.

LEMMA 3. *A necessary condition for $v \in CB(3, \lambda)$ is that*
(i) $\lambda \equiv 1, 5, 7, 11 \pmod{12}$ *and* $v \equiv 1, 3 \pmod 6$ *or*
(ii) $\lambda \equiv 2, 10 \pmod{12}$ *and* $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ *or*
(iii) $\lambda \equiv 3, 9 \pmod{12}$ *and* $v \equiv 1 \pmod 2$ *or*
(iv) $\lambda = 4, 8 \pmod{12}$ and $v = 0, 1 \pmod 3$ *or*
(v) $\lambda \equiv 6 \pmod{12}$ *and* $v \equiv 0, 1, 3 \pmod 4$ *or*
(vi) $\lambda \equiv 0 \pmod{12}$ *and* $v \geqslant 3$.

In the remainder of this paper, we demonstrate that this necessary condition is sufficient with precisely two exceptions. In order to do this, we use the fact $CB(k, \lambda) \subseteq CB(k, n\lambda)$ for any integer $n \geq 1$. This is easy; one can construct a $CB[k, n\lambda; v]$ from a $CB[k, \lambda; v]$ B by simply taking each block of B n times. A consequence of this remark is that it suffices to consider $\lambda = 1, 2, 3, 4, 6$ or 12.

A significant amount of research has been carried out on the existence of $CB[3, 1; v]$ designs by Netto [16], Heffter [9], Peltesohn [19], Skolem [22], Hanani [6], O'Keefe [18], Rosa [21] and Hilton [10]. We refer the reader to [4] for a survey of this work. Peltesohn [19] showed the following theorem.

THEOREM 4. *$v \in CB(3, 1)$ if and only if $v = 1, 3 \pmod 6$ and $v \neq 9$.*

3. THE EXISTENCE OF $CB[3, \lambda; v]$

We now use a simple proof technique to examine the existence of $CB[3, \lambda; v]$ for $\lambda > 1$.

LEMMA 5. *$v \in CB(3, 2)$ if and only if $v \equiv 0, 1, 3, 4, 7, 9 \pmod{12}$ and $v \neq 9$.*

PROOF. For $v \equiv 1, 3 \pmod 6$, taking two copies of Peltesohn's $CB[3, 1; v]$ design gives a $CB[3, 2; v]$ design. The only omission here is a $CB[3, 2; 9]$ design; a short argument demonstrates that $9 \notin CB(3, 2)$. It remains only to consider $v \equiv 0, 4 \pmod{12}$.

CLAIM 5.1. *$12m \in CB(3, 2)$ for all $m \geq 1$.*

PROOF. Consider the difference triples

$$\begin{array}{ll}
(2r+1, 5m-r, 5m+r+1) & 0 \leq r \leq m-1 \\
(2m-2r-2, 2m+r+1, 4m-r-1) & 0 \leq r \leq m-2 \\
(2m-2r-1, 2m+r, 4m-r-1) & 0 \leq r \leq m-1 \\
(2r+2, 5m-r-1, 5m+r+1) & 0 \leq r \leq m-2 \\
(2m, 3m, 5m). &
\end{array}$$

These triples partition $D_2(12m, 2)$.

CLAIM 5.2. *$12m + 4 \in CB(3, 2)$ for $m \geq 0$.*

PROOF. Consider the difference triples

$$\begin{array}{lll}
(2r+2, 3m-r, 3m+r+2) & 0 \leq r \leq m-1, & \text{taken twice} \\
(2m-2r-1, 4m+r+2, 6m-r+1) & 0 \leq r \leq m-1, & \text{taken twice} \\
(3m+1, 3m+1, 6m+2). & &
\end{array}$$

These difference triples partition $D(12m+4, 2)$.

We conclude that for $v \equiv 0, 4 \pmod{12}$ $v \in CB(3, 2)$; this completes the proof of the lemma.

In the case $\lambda = 3$, Hwang and Lin's determination of $B(3, 3)$ [12] also determines $CB(3, 3)$. We present a simpler proof here, suggested by one of the referees.

LEMMA 6. *$v \in CB(3, 3)$ if and only if $v \equiv 1 \pmod 2$.*

PROOF. $\{(r, r, \min(2r, 2m+1-2r)), 1 \leq r \leq m\}$ partitions $D(2m+1, 3)$ when $(2m+1) \not\equiv (\text{mod } 3)$. For $(2m+1) \equiv 0 (\text{mod } 3)$, $\{(r, r, \min(2r, 2m+1-2r)), \quad 1 \leq r \leq m, \quad r \neq (2m+1)/3\}$ partitions $D_3(2m+1, 3)$.

LEMMA 7. *$v \in CB(3, 4)$ if and only if $v \equiv 0, 1 (\text{mod } 3)$.*

PROOF. Duplicating the designs constructed in Lemma 6 gives $CB[3, 4; v]$ for all $v \equiv 0, 1, 3, 4, 7, 9 (\text{mod } 12)$ with the exception of $v = 9$. A $CB[3, 4; 9]$ exists with starter blocks $\{\{0, 1, 3\}, \{0, 1, 2\}, \{0, 2, 5\}, \{0, 2, 5\}, \{0, 1, 5\}, \{0, 3, 6\}\}$. We therefore need only consider $v \equiv 6, 10 (\text{mod } 12)$.

CLAIM 7.1. *$12m + 6 \in CB(3, 4)$ for $m \geq 1$.*

PROOF. Consider the following difference triples:

$$\begin{array}{ll}
(2r+1, 3m-r+1, 3m+r+2) & 0 \leq r \leq m-1 \\
(2r+2, 5m-r+2, 5m+r+4) & 0 \leq r \leq m-2 \\
(2r+2, 3m-r-1, 3m+r+1) & 0 \leq r \leq m-2 \\
(2r+1, 5m-r+2, 5m+r+3) & 0 \leq r \leq m-1 \\
(2r+2, 3m-r+1, 3m+r+3) & 0 \leq r \leq m-2 \\
(2r+1, 5m-r+3, 5m+r+4) & 0 \leq r \leq m-1 \\
(2r+2, 3m-r, 3m+r+2) & 0 \leq r \leq m-2 \\
(2r+1, 5m-r+2, 5m+r+3) & 0 \leq r \leq m-1 \\
(2m, 2m+1, 4m+1) & \text{taken twice} \\
(3m+1, 3m+2, 6m+3) & \\
(2m+1, 2m+2, 4m+3) & \\
(2m, 2m, 4m) & \\
(3m, 4m+3, 5m+3). &
\end{array}$$

These triples partition $D_4(12m+6, 4)$.
A $CB[3, 4; 6]$ has starter blocks $\{0, 1, 2\}$, $\{0, 1, 3\}$, $\{0, 1, 3\}$ and $\{0, 2, 4\}$.

CLAIM 7.2. *$12m + 10 \in CB(3, 4)$ for $m \geq 0$.*

PROOF. Consider the following difference triples:

$$\begin{array}{ll}
(2r+1, 3m-r+3, 3m+r+4) & 0 \leq r \leq m-1 \\
(2r+2, 5m-r+4, 5m+r+6) & 0 \leq r \leq m-1 \\
(2r+1, 3m-r+2, 3m+r+3) & 0 \leq r \leq m-1 \\
(2r+2, 5m-r+3, 5m+r+5) & 0 \leq r \leq m-1 \\
(2r+2, 3m-r, 3m+r+2) & 0 \leq r \leq m-1 \\
(2r+1, 5m-r+5, 5m+r+6) & 0 \leq r \leq m-1 \\
(2r+2, 3m-r+1, 3m+r+3) & 0 \leq r \leq m-1 \\
(2r+1, 5m-r+3, 5m+r+4) & 0 \leq r \leq m-1 \\
(2m+1, 2m+3, 4m+4) & \\
(2m+1, 2m+2, 4m+3) & \\
(4m+2, 4m+3, 4m+5) & \\
(2m+1, 4m+3, 6m+4) & \\
(3m+1, 4m+4, 5m+5) & \\
(2m+2, 3m+2, 5m+4). &
\end{array}$$

These triples partition $D(12m+10, 4)$.

The proofs of these two claims complete the proof of Lemma 7.

LEMMA 8. *$v \in CB(3,6)$ if and only if $v \equiv 0, 1, 3 \pmod 4$.*

PROOF. For $v \equiv 1, 3 \pmod 4$, Lemma 6 supplies the necessary constructions. For $v \equiv 0, 4 \pmod{12}$ the necessary construction is given by Lemma 5. We thus need only consider $v \equiv 8 \pmod{12}$; we give a more general proof for $v \equiv 0 \pmod 4$.

CLAIM 8.1. *$4m \in CB(3,6)$ for all $m \geq 0$.*

PROOF. Consider the following difference triples:

$$\begin{array}{ll}(2r-1, 2r, \min(4r-1, 4m-4r+1)) & 1 \leq r \leq m, \text{ taken twice} \\ (r, r, \min(2r, 4m-2r)) & 1 \leq r \leq 2m-1.\end{array}$$

These triples partition $D(4m, 6)$.

The proof of the claim completes the proof of Lemma 8.

LEMMA 9. *$v \in CB(3,12)$ if and only if $v \geq 3$.*

PROOF. By Lemma 8 we need only consider $v \equiv 2 \pmod 4$ and by Lemma 7 we need only consider $v \equiv 2 \pmod 3$; hence, we need only consider $v \equiv 2 \pmod{12}$.

CLAIM 9.1. *$12m+2 \in CB(3,12)$ for $m \geq 1$.*

PROOF. Consider the following difference triples:

$$\begin{array}{lll}(2r+2, 3m-r, 3m+r+2) & 0 \leq r \leq m-2, & \text{taken six times} \\ (2r+1, 5m-r+1, 5m+r+2) & 0 \leq r \leq m-1, & \text{taken six times} \\ (2r+1, 3m-r, 3m+r+1) & 0 \leq r \leq m-1, & \text{taken twice} \\ (2r+2, 5m-r, 5m+r+2) & 0 \leq r \leq m-2, & \text{taken twice} \\ (2r+1, 3m-r+1, 3m+r+2) & 0 \leq r \leq m-1, & \text{taken twice} \\ (2r+2, 5m-r, 5m+r+2) & 0 \leq r \leq m-2, & \text{taken twice} \\ (2r+1, 3m-r, 3m+r+1) & 0 \leq r \leq m-1 & \\ (2r+2, 5m-r, 5m+r+2) & 0 \leq r \leq m-2 & \\ (2r+1, 3m-r-1, 3m+r) & 0 \leq r \leq m-1 & \\ (2r+2, 5m-r-1, 5m+r+1) & 0 \leq r \leq m-2 & \\ (2m, 2m+1, 4m+1) & \text{taken seven times} & \\ (2m, 3m+1, 5m+1) & \text{taken four times} & \\ (2m+1, 4m+1, 6m) & & \\ (3m+1, 4m, 5m+1) & & \\ (3m+1, 4m+1, 5m). & & \end{array}$$

These triples partition $D(12m+2, 12)$.

The proof of this claim completes the proof of Lemma 9.

We have at this point almost resolved the existence of $CB[3, \lambda; v]$. We have only to resolve whether $9 \in CB(3,5)$. In conclusion then, a $CB[3,5;9]$ has the starter blocks $\{\{0,1,2\}, \{0,1,3\}, \{0,1,4\}, \{0,1,5\}, \{0,2,4\}, \{0,2,5\}, \{0,3,6\}, \{0,3,6\}\}$. We have in our progression of lemmas proved the following theorem.

THEOREM 10. *The necessary condition for $v \in CB(3, \lambda)$, which is that*
(i) $\lambda \equiv 1, 5, 7, 11 (\mathrm{mod}\ 12)$ *and* $v \equiv 1, 3 (\mathrm{mod}\ 6)$ *or*
(ii) $\lambda \equiv 2, 10 (\mathrm{mod}\ 12)$ *and* $v \equiv 0, 1, 3, 4, 7, 9 (\mathrm{mod}\ 12)$ *or*
(iii) $\lambda \equiv 3, 9 (\mathrm{mod}\ 12)$ *and* $v \equiv 1 (\mathrm{mod}\ 2)$ *or*
(iv) $\lambda \equiv 4, 8 (\mathrm{mod}\ 12)$ *and* $v \equiv 0, 1 (\mathrm{mod}\ 3)$ *or*
(v) $\lambda \equiv 6 (\mathrm{mod}\ 12)$ *and* $v \equiv 0, 1, 3 (\mathrm{mod}\ 4)$ *or*
(vi) $\lambda \equiv 0 (\mathrm{mod}\ 12)$ *and* $v \geqslant 3$,
is also sufficient with only two exceptions; $9 \notin CB(3, 1)$ *and* $9 \notin CB(3, 2)$.

ACKNOWLEDGEMENTS

We would like to thank the referees for their valuable comments.

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Received 10 December 1979 and in revised form 21 April 1980

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